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ON THE BEST POSITION OF ELASTIC SYMMETRY PLANES IN AN ORTHOTROPIC BODY*

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A problem of optimal positioning of elastic symmetry planes in an orthotropic body is considered. The criterion used is that of the minimum of potential energy of deformation characterizing the rigidity of the body. Analysis of the conditions of optimality leads to the interchangeability of the stress and deformation tensors. Mechanically different methods of realizing these conditions are established. An analogous analysis was carried out in /1/ for two-dimensional problems.

1. Consider a linear elastic body V, bounded by the surface S. The elastic state of this body is described by the displacement vector $\mathbf{u}$, strain ε and stress σ tensors, all connected by the relations /2/

$$\operatorname{div}\sigma + \mathbf{F} = 0,\quad 2\varepsilon = \nabla\mathbf{u} + (\nabla\mathbf{u})^T \tag{1.1}$$

in which $\mathbf{F}$ is the volume force vector and the superscript T denotes transposition. The elasticity relations are defined by the fourth rank elasticity tensor, and have the form /3/

$$\varepsilon = \mathbf{L}\cdot\cdot\sigma \tag{1.2}$$

The boundary conditions are

$$\mathbf{u} = 0 \text{ on } S_1,\quad \mathbf{n}\cdot\sigma = \mathbf{T} \quad \text{on } S_2 \tag{1.3}$$

where $\mathbf{n}$ is the unit vector of the outward normal directed towards the body and $\mathbf{T}$ is a given vector of surface forces. The potential energy of deformation Π and work A done by external forces are connected by the formula /2/

$$\Pi = \frac{1}{2}A = \frac{1}{2}\int\limits_V \sigma\cdot\cdot\mathbf{L}\cdot\cdot\sigma\, dV \tag{1.4}$$

Let us denote by $\mathbf{e}_n$ the principal directions of elasticity, i.e. the unit vectors perpendicular to the elastic symmetry planes of the body, and by $\mathbf{A}$ the rotation tensor connecting the unit vectors $\mathbf{e}_n$ with the unit vectors of some fixed $\mathbf{r}_n$ coordinate system satisfying the relations

$$\mathbf{e}_n = \mathbf{A}\cdot\mathbf{r}_n,\quad \mathbf{A}^T\cdot\mathbf{A} = \mathbf{E} \tag{1.5}$$

where $\mathbf{E}$ is a unit tensor of second rank. Expanding the tensor $\mathbf{L}$ over a basis of the fourth rank tensors, we obtain

$$\mathbf{L} = L^\circ_{mnpq}\mathbf{A}\cdot\mathbf{r}_m \otimes \mathbf{A}\cdot\mathbf{r}_n \otimes \mathbf{A}\cdot\mathbf{r}_p \otimes \mathbf{A}\cdot\mathbf{r}_q \tag{1.6}$$

where the summation is carried out over the repeated indices. We note that the components L°_{mnpq} of the tensor $\mathbf{L}$ can be expressed on the basis of the principal stresses in terms of the "technical" constants according to the formulas /3/

$$\begin{gathered}
L^\circ_{mnpq} = L^\circ_{nmpq} = L^\circ_{mnqp} = L^\circ_{pqmn}\\
L^\circ_{1111} = \frac{1}{E_1},\quad L^\circ_{2222} = \frac{1}{E_2},\quad L^\circ_{3333} = \frac{1}{E_3},\quad L^\circ_{1122} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}\\
L^\circ_{2233} = -\frac{\nu_{23}}{E_2} = -\frac{\nu_{32}}{E_3},\quad L^\circ_{3311} = -\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1},\quad L^\circ_{1212} = \frac{1}{2G_{12}},\\
L^\circ_{2323} = \frac{1}{2G_{23}},\quad L^\circ_{3131} = \frac{1}{2G_{31}}
\end{gathered} \tag{1.7}$$

Here E_k and G_{km} are the Young's and shear moduli and ν_{km} are the Poisson's ratios ($k \neq m$, $k, m = 1, 2, 3$).

We consider the following problem of optimization. We require to find the rotation tensor $\mathbf{A}$ imparting an extremal value to the functional (1.4), with the relations (1.1)—(1.3),

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(1.5) — (1.7) fulfilled.

Let us construct a new functional

$$\Phi = \Pi + \int_V [\mathbf{u}\cdot(\nabla\cdot\sigma + \mathbf{F}) + \gamma\cdot\cdot(\mathbf{A}^T\cdot\mathbf{A} - \mathbf{E})]\, dV$$

which coincides with the old functional when all constraints are fulfilled. Here γ is a symmetric tensor of second rank, serving here as a Lagrange multiplier. Having computed the first variation of this functional, we use the Ostrogradskii formula and equations (1.1) — (1.3) to obtain

$$\delta\Phi = \int_V \left[\frac{1}{2}\sigma\cdot\cdot\delta\mathbf{L}\cdot\cdot\sigma + 2\gamma\cdot\cdot(\mathbf{A}\cdot\delta\mathbf{A}^T)\right] dV \geqslant 0$$

Varying the formula (1.6), we obtain

$$\delta\mathbf{L} = \mathring{L}_{mnpq}(\delta\mathbf{A}\cdot\mathbf{r}_m\otimes\mathbf{e}_n\otimes\mathbf{e}_p\otimes\mathbf{e}_q + \mathbf{e}_m\otimes\delta\mathbf{A}\cdot\mathbf{r}_n\otimes\mathbf{e}_p\otimes\mathbf{e}_q + \mathbf{e}_m\otimes\mathbf{e}_n\otimes\delta\mathbf{A}\cdot\mathbf{r}_n\otimes\mathbf{e}_q + \mathbf{e}_m\otimes\mathbf{e}_n\otimes\mathbf{e}_p\otimes\delta\mathbf{A}\cdot\mathbf{r}_q)$$

Carrying out the lengthy, although elementary transformations, we obtain the final expression for the first variation

$$\delta\Phi = \int_V \delta\mathbf{A}\cdot\cdot(\mathbf{A}^T\cdot\varepsilon\cdot\sigma + \gamma\cdot\mathbf{A}^T)dV \geqslant 0 \tag{1.8}$$

Choosing suitably the components of the symmetric tensor γ, we can make the coefficients accompanying six variations of the components of the tensor $\mathbf{A}$ vanish. The three remaining coefficients must be equal to zero by virtue of the arbitrary character of the variations. Taking into account the symmetry of the tensors γ, ε and σ, we finally obtain the optimality condition

$$\varepsilon\cdot\sigma = \sigma\cdot\varepsilon \tag{1.9}$$

which yields three equations for determining three unknown components of the tensor $\mathbf{A}$.

If we denote by $\sigma_{km}^\circ, \varepsilon_{km}$ the components of the tensors σ and ε on the basis $\mathbf{e}_n$ and take into account the relations (1.2) and (1.7), the above three equations will become

$$\begin{gathered}\mathring{\sigma}_{23}A_1 + \mathring{\sigma}_{31}\mathring{\sigma}_{12}(1/G_{12} - 1/G_{31}) = 0 \quad (1\to 2\to 3)\\ \left(A_1 = \mathring{\sigma}_{11}\frac{\nu_{13}-\nu_{12}}{E_1} + \mathring{\sigma}_{22}\left(\frac{1+\nu_{23}}{E_2} - \frac{1}{G_{23}}\right) - \mathring{\sigma}_{33}\left(\frac{1+\nu_{32}}{E_3} - \frac{1}{G_{23}}\right)(1\to 2\to 3)\right.\end{gathered} \tag{1.10}$$

(the remaining relations can be obtained by circular interchange of the indices 1,2,3).

2. In the general case when all Young's moduli and shear moduli are different, the relations (1.10) imply the existence of three types of zones (i.e. the methods of realizing the conditions of stationarity). In the first type zones the following relations hold:

$$\sigma_{km}^\circ = 0,\ k \neq m,\quad k,\ m = 1,2,3$$

and this means that the principal directions of the elasticity coincide with the principal directions of the tensor σ. In the second type zones one of the following relations holds:

$$\begin{gathered}\sigma_{31}^\circ = \sigma_{23}^\circ = 0,\quad A_3 = 0,\quad \sigma_{12}^\circ \neq 0\\ \sigma_{12}^\circ = \sigma_{31}^\circ = 0,\quad A_1 = 0,\quad \sigma_{23}^\circ \neq 0\\ \sigma_{23}^\circ = \sigma_{12}^\circ = 0,\quad A_2 = 0,\quad \sigma_{31}^\circ \neq 0\end{gathered}$$

and here only one principal direction of the elasticity coincides with a principal direction of the tensor σ. In the third type zones all shear stresses σ_{km}° are different from zero.

Let us analyze the relations (1.10) for the case when the principal values of $\sigma_1, \sigma_2, \sigma_3$ are all different. The commutativity of the tensors ε and σ implies their coaxiality, therefore we can write the elasticity relations (1.2) in the form /2/

$$\varepsilon = \varphi_0\mathbf{E} + \varphi_1\sigma + \varphi_2\sigma^2 \tag{2.1}$$

where $\varphi_0, \varphi_1, \varphi_2$ are known functions of $\sigma_1, \sigma_2, \sigma_3$, and hence of the principal invariants of the tensor σ. The above functional dependence is determined by a system of algebraic equations of the form

$$\varepsilon_k(\sigma_1, \sigma_2, \sigma_3) = \varphi_0 + \varphi_1\sigma_k + \varphi_2\sigma_k^2\ (k = 1, 2, 3) \tag{2.2}$$

where ε_k are the principal values of the tensor ε. If the coefficients $\varphi_0, \varphi_1, \varphi_2$ have been found, then the optimal state is determined from the equations (1.1), (1.3) and (2.1), and the optimal rotation tensor $\mathbf{A}$ is found from (1.10) in terms of the known tensor σ.

The form of the functions $\varepsilon_k(\sigma_1, \sigma_2, \sigma_3)$ depends on the type of the zone. We shall describe a general method of finding these functions and indicate some particular features of the different types of zones. Let us denote by $\mathbf{r}_k^\circ$ the principal directions of the tensor σ and by L_{mnpq} the components of the elasticity tensor on the basis $\mathbf{r}_k^\circ$. Since the tensors ε and σ are coaxial, we have

$$L_{km11}\sigma_1 + L_{km22}\sigma_2 + L_{km33}\sigma_3 = 0 \quad (k \neq m, \quad k, m = 1, 2, 3)$$
$$L_{mnpq} = L_{rstk}^\circ \alpha_{mr}\alpha_{ns}\alpha_{pt}\alpha_{qk}, \quad \alpha_{km}\alpha_{kn} = \delta_{mn} \quad (\mathbf{e}_r = \alpha_{mr}\mathbf{r}_m^\circ)$$

The above system yields the coefficients α_{mr} in terms of σ_k, L_{rstk}°. In this manner we can obtain the relationship $L_{mnpq}(\sigma_1, \sigma_2, \sigma_3)$ and hence the functions in question (no summation to be performed over k!)

$$\varepsilon_k = L_{kk11}(\sigma_1, \sigma_2, \sigma_3)\,\sigma_1 + L_{kk22}(\sigma_1, \sigma_2, \sigma_3)\,\sigma_2 + L_{kk33}(\sigma_1, \sigma_2, \sigma_3)\,\sigma_3, \quad (k = 1, 2, 3) \tag{2.3}$$

In the first zone we have

$$\alpha_{mn} = \delta_{mn}, \quad \varepsilon_1 = \frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3 \quad (1 \to 2 \to 3)$$

The tensor $\mathbf{A}$ is found by reducing the known tensor σ to its principal axes. Let us put $\sigma_+ = \max(\sigma_1, \sigma_2, \sigma_3)$, $\sigma_- = \min(\sigma_1, \sigma_2, \sigma_3)$. This yields the first type zones: 1) $\sigma_1 = \sigma_+$, $\sigma_3 = \sigma_-$, 2) $\sigma_1 = \sigma_-$, $\sigma_2 = \sigma_+$ etc. Therefore the maximum possible number of the first type zones is six.

As an example of the zone of the second type, we shall consider the zones for which the following relations hold:

$$\sigma_{13}^\circ = \sigma_{23}^\circ = 0,\ A_3 = 0,\ \sigma_{12}^\circ \neq 0,\ \sigma_{33}^\circ = \sigma_3 \tag{2.4}$$
$$(\alpha_{11} = \alpha_{22} = \cos\varphi, \quad \alpha_{21} = -\alpha_{12} = \sin\varphi, \quad \alpha_{31} = \alpha_{32} = \alpha_{13} = \alpha_{23} = 0,$$
$$\alpha_{33} = 1, \quad \mathbf{e}_3 = \mathbf{r}_3^\circ)$$

We note that the equation $A_3 = 0$ determines two values of φ for the given tensor σ. Indeed,

$$\cos 2\varphi = \left(a\sigma_3 + b\,\frac{\sigma_1 + \sigma_2}{2}\right)\left(\frac{\sigma_1 - \sigma_2}{2}\right)^{-1}$$

$$a = -\frac{\nu_{32} - \nu_{31}}{E_3}\left(\frac{1+\nu_{21}}{E_2} + \frac{1+\nu_{12}}{E_1} - \frac{2}{G_{12}}\right)^{-1}, \quad b = -\left(\frac{1}{E_1} - \frac{1}{E_2}\right) \times \left(\frac{1+\nu_{12}}{E_1} + \frac{1+\nu_{21}}{E_2} - \frac{2}{G_{12}}\right)^{-1}$$

The formulas (2.3) can be written as

$$\varepsilon_1 + \varepsilon_2 = \frac{\sigma_1 + \sigma_2}{2}\left(\frac{1-\nu_{12}}{E_1} + \frac{1-\nu_{21}}{E_2}\right) - \sigma_3\,\frac{\nu_{31} + \nu_{32}}{E_3} + \left(\frac{1}{E_1} - \frac{1}{E_2}\right) \times \left(\sigma_3 a + \frac{\sigma_1 + \sigma_2}{2}\,b\right)$$

$$\varepsilon_1 - \varepsilon_2 = \frac{\sigma_1 - \sigma_2}{G_{12}}, \quad \varepsilon_3 = -\frac{\sigma_1 + \sigma_2}{2}\,\frac{\nu_{32} + \nu_{31}}{E_3} + \frac{\sigma_3}{E_3} + \frac{\nu_{32} - \nu_{31}}{E_3} \times \left(\sigma_3 a + \frac{\sigma_1 + \sigma_2}{2}\,b\right)$$

To determine the rotation tensor $\mathbf{A}$, we must reduce the tensor σ to its principal axes and superimpose the principal direction of the elasticity $\mathbf{e}_3$ on one of the principal directions of the tensor σ. The remaining two are obtained by rotating the other two principal directions by the angle φ about the unit vector $\mathbf{e}_3$. Thus the relations (2.4) define not more than twelve zones. Since we can repeat the above procedure with the remaining two principal directions of the elasticity, $\mathbf{e}_1$ and $\mathbf{e}_2$, it follows that the number of zones of the second type does not exceed thirty six.

The most complicated determination of the functions $\varepsilon_k(\sigma_1, \sigma_2, \sigma_3)$ is encountered in the case of the zones of the third type. Here, if the "technical" constants are not connected to each other by some special relationship, then a general construction must be used. The same procedure can be adopted when $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are all different. In this case the tensors ε and σ in (2.1) must be interchanged. When the principal values contain pairs of identical values (e.g. $\sigma_1 \neq \sigma_2 = \sigma_3$, $\varepsilon_2 = \varepsilon_3$), then one of the coefficients φ_0, φ_1 and φ_2 in (2.1) can be made equal to zero and the relation $\varepsilon_k(\sigma_1, \sigma_2)$ sought. The case $\varepsilon_1 \neq \varepsilon_2 = \varepsilon_3$, $\sigma_2 = \sigma_3$ can be dealt with in the same manner, and by continuing this procedure we can go through all possible variants of the states of stress.

Let us consider a particular case when the body has cubic symmetry

$$E_1 = E_2 = E_3 = E, \ G_{12} = G_{23} = G_{31} = G, \ \nu_{13} = \nu_{31} = \nu_{23} = \nu_{32} = \nu_{21} = \nu_{12} = \nu$$

Here we have a single zone of the first type in which

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\mathrm{E}\,\mathrm{Tr}\,\sigma, \quad \overset{\circ}{\sigma}_{km} = 0 \quad (k \neq m)$$

and one zone of the third type

$$\varepsilon = \frac{1}{G}\sigma - \frac{1}{3}\left(\frac{1}{G} - \frac{1-2\nu}{E}\right)\mathrm{E}\,\mathrm{Tr}\,\sigma, \quad \overset{\circ}{\sigma}_{11} = \overset{\circ}{\sigma}_{22} = \overset{\circ}{\sigma}_{33}$$

The maximum number of zones of the second type is three, and we have

$$\varepsilon_{1,2} = \frac{\sigma_1+\sigma_2}{2}\frac{1-\nu}{E} - \frac{\nu}{E}\sigma_3 \pm \frac{1}{2G}(\sigma_1-\sigma_2), \quad \varepsilon_3 = -\frac{\nu}{E}(\sigma_1+\sigma_2) + \frac{\sigma_3}{E}, \quad \varphi = \pi/4, \quad \overset{\circ}{\sigma}_{13} = \overset{\circ}{\sigma}_{23} = 0$$

In the first zone $\sigma_3 = \sigma_-$, in the second zone $\sigma_3 = \sigma_+$ and in the third zone σ_3 is equal to the intermediate principal value of the tensor σ.

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